

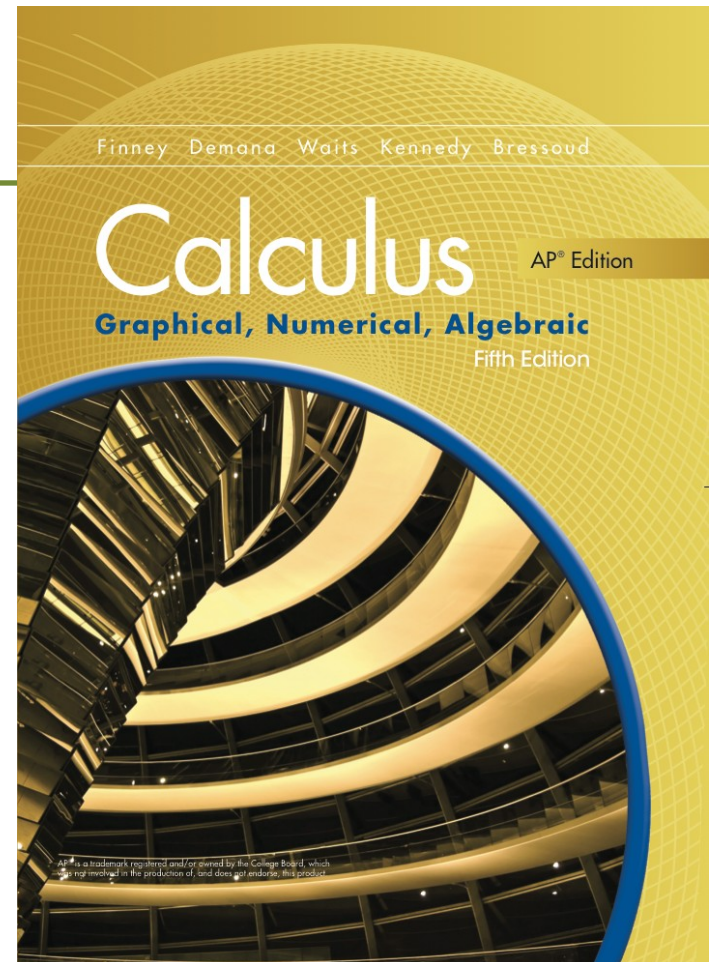
Chapter 8

Applications of Definite Integral



Section 8.2

Areas in the Plane



Quick Review

1. Find the area between the x -axis and the graph of the given function over the given interval.

a. $y = \cos x$ over $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

b. $y = e^x$ over $[0, 1]$

c. $y = 3x^2 + 4x + 1$ over $[-1, 1]$

2. Find the x - and y - coordinates of all points where the graphs of the given functions intersect.

a. $y = x^2$ and $y = x + 2$

b. $y = e^x$ and $y = x + 1$

c. $y = \cos x$ and $y = x^3$

Quick Review Solutions

1. Find the area between the x -axis and the graph of the given function over the given interval.

a. $y = \cos x$ over $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ 2

b. $y = e^x$ over $[0, 1]$ e^1

c. $y = 3x^2 + 4x + 1$ over $[-1, 1]$ 4

2. Find the x - and y - coordinates of all points where the graphs of the given functions intersect.

a. $y = x^2$ and $y = x + 2$ $(-1, 1)$ and $(2, 4)$

b. $y = e^x$ and $y = x + 1$ $(0, 1)$

c. $y = \cos x$ and $y = x^3$ $(0.865, 0.648)$

What you'll learn about

- Areas as limits of Riemann sums
- Areas between curves
- Areas for which integration is with respect to y

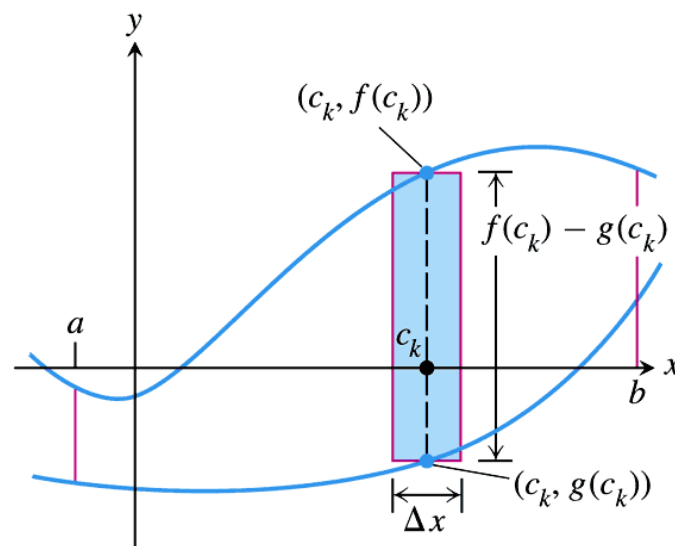
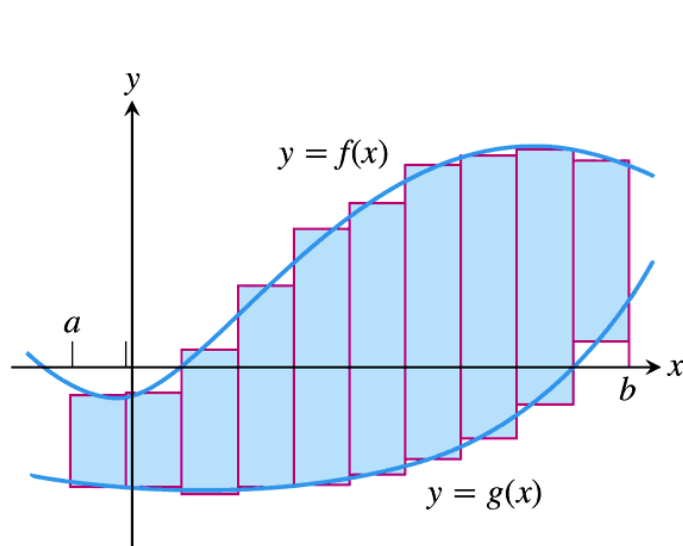
...and why

The techniques of this section allow us to compute areas of complex regions of the plane.

Area Between Curves

Partition the region into vertical strips of equal width Δx

Each rectangle has area $[f(c_k) - g(c_k)] \Delta x$ for some c_k in its respective subinterval. Approximate the area of each region with the Riemann sum $\sum [f(c_k) - g(c_k)] \Delta x$



The limit of these sums as $\Delta x \rightarrow 0$ is $\int_a^b [f(x) - g(x)] dx$

Area Between Curves

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of $[f - g]$ from a to b ,

$$A = \int_a^b [f(x) - g(x)] dx$$

Example Applying the Definition

Find the area of the region between $y = \cos x$ and $y = \sin x$

from $x = 0$ to $x = \frac{\pi}{3}$.

Note that $y = \cos x$ is above $y = \sin x$ in the given interval.

$$\begin{aligned} A &= \int_0^{\frac{\pi}{3}} [\cos x - \sin x] dx \\ &= \left[\sin x + \cos x \right]_0^{\frac{\pi}{3}} \\ &= \frac{\sqrt{3} - 1}{2} \text{ units squared.} \end{aligned}$$

Example Area of an Enclosed Region

Find the area of the region enclosed by the parabola $y = x^2 - 1$ and $y = x + 1$.

Graph the curves to view the region.

The limits of integration are found by solving the equation $x^2 - 1 = x + 1$.

The solutions are $x = -1$ and $x = 2$.

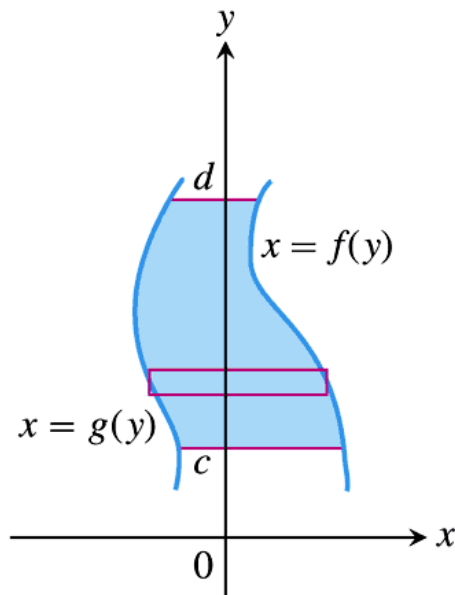
Since the line lies above the parabola on $[-1, 2]$, the area integrand is $x + 1 - (x^2 - 1)$.

$$\begin{aligned} A &= \int_{-1}^2 [-x^2 + x + 2] \, dx \\ &= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_{-1}^2 \\ &= \frac{9}{2} \text{ units squared.} \end{aligned}$$

Integrating with Respect to y

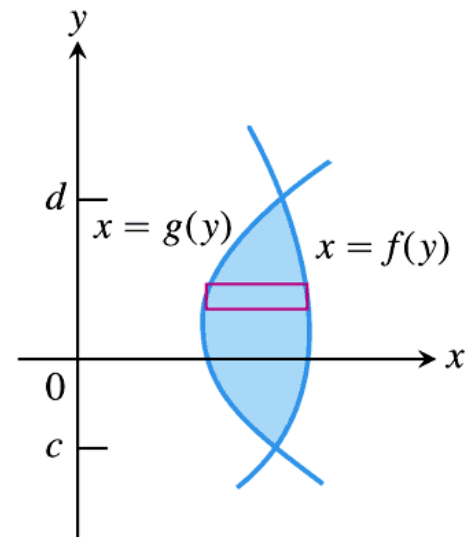
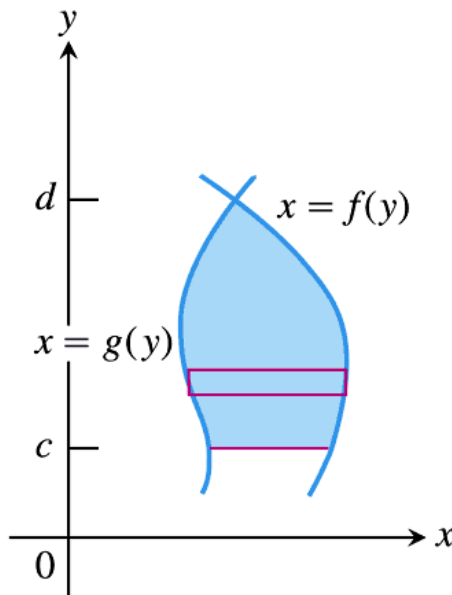
If the boundaries of a region are more easily described by functions of y , use horizontal approximating rectangles.

For regions like these



use this formula

$$A = \int_c^d [f(y) - g(y)] dy.$$



Example Integrating with Respect to y

Find the area of the region bounded by the curves $x = y^2 - 1$ and $y = x + 1$.

Solve for x in terms of y : $x = y^2 - 1$ and $x = y - 1$. Find the points of intersection: $(-1, 0)$ and $(0, 1)$.

$$\begin{aligned} A &= \int_0^1 [(y - 1) - (y^2 - 1)] dy \\ &= \int_0^1 [y - y^2] dy \\ &= \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 \\ &= \frac{1}{6} \end{aligned}$$

Example Using Geometry

Find the area of the region enclosed by the graphs of $y = \sqrt{x+1}$, $y = x - 1$ and the x -axis.

Find the area under the curve $y = \sqrt{x+1}$ over the interval $[-1, 3]$ and subtract the area of the triangle:

$$A = \int_{-1}^3 (\sqrt{x+1}) dx - \frac{1}{2}(2)(2)$$

$$= \frac{2}{3} (x+1)^{\frac{3}{2}} \Big|_{-1}^3 - 2$$

$$= \frac{10}{3} \text{ units squared}$$